Definition of group: a set G equipped with an operation \( \cdot \) that combines any two elements \( a, b \) to form another element \( a \cdot b \). 

\( (G, \cdot) \) must satisfy:

1) closure \( \forall a, b \in G, a \cdot b \in G \)
2) associativity \( \forall a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c) \)
3) identity there exists an element \( e \in G : \forall a \in G, e \cdot a = a \cdot e = a \)
4) inverse \( \forall a \in G \) there exists \( b \in G \) such that \( a \cdot b = b \cdot a = e \) 

\( (b \) is denoted \( a^{-1} \) \)

A Lie group is a group with \( \infty \) number of elements that is also a differentiable manifold. Any group element continuously connected with 11 can be written as

\[ U = e^{\gamma a^a T^a} \]

where \( a^a \) are numbers and \( T^a \) are the group generators.

If we know the explicit form of the group elements \( U \), we can deduce the form of the \( T^a \) by expanding around 11.

For example: orthochronous Lorentz, \( SO(1,3) \)

boost along \( x^2 \) axis is

\[ \Lambda^H U = \begin{pmatrix} 1 & \beta & 0 & 0 \\ \beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ \gamma = \frac{1}{\sqrt{1-\beta^2}} \]

\[ x^1 = \gamma (t + \beta x) \]
\[ t^1 = \gamma (t + \beta x) \]
\[ y^1 = y \]
\[ z^1 = z \]

at \( o(\beta) \),

\[ \Lambda^H U = \begin{pmatrix} 1 & \beta & 0 & 0 \\ \beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta^H U + \beta \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \]

now

\[ U = 1 + \alpha^a T^a \]

so \( \alpha^a \to \beta \)

and \( \alpha^a T^a \to \omega^H U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)

\( \text{generator of boosts along } x^2 \)
The generators $T^a$ form a Lie algebra, defined through the commutation relations

$$[T^a, T^b] = i \delta^{abc} T^c$$

structure constants

the group is

- Abelian if $\delta^{abc} = 0$
- Non-Abelian otherwise e.g. su(2) has $\delta^{abc} = \epsilon^{abc}$ totally antisymmetric with $\epsilon^{123} = 1$

note that $[A, B] = AB - BA$


and so

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

Jacobi identity

Now in terms of structure constants, $\{ A \to a$, $B \to b$, $C \to c$

$$[T^a, [T^b, T^c]] = [T^a, \delta^{bcd} T^d] = \delta^{bcd} [T^a, T^d] = \delta^{bcd} \delta^{e} \delta^{ede} T^e$$

then $\circledast$ is

$$(- \delta^{bad} \delta^{ede} - \delta^{cad} \delta^{bde} - \delta^{abd} \delta^{cde}) T^e = 0$$

$$\delta^{bad} \delta^{ede} + \delta^{cad} \delta^{bde} + \delta^{abd} \delta^{cde} = 0$$

Jacobi id, for structure constants

The comm. relations completely determine structure of the group sufficiently close to 1 if we go far away then global aspects matter (e.g. su(2) and o(3) which have same comm. relations but differ globally).

But this is not relevant for introductory description of non-Abelian gauge theories.

Note also that $\delta^{abc}$ are antisymmetric in first 2 indices:

$$[T^a, T^b] = \delta^{abc} T^c$$

$$- [T^b, T^a] = - \delta^{bac} T^c$$

$$\Rightarrow \delta^{abc} = - \delta^{bac}.$$
An ideal is a sub-algebra \( I \subset g \) such that
\[ [g, n] \in \equiv I \quad \text{for any} \quad \{ g \in g, n \in I \} \quad \text{(invariant subalgebra)} \]

0 and the whole \( g \) are trivial ideals; an algebra that does not admit a non-trivial ideal is a simple algebra.

\[ \text{e.g.} \quad \text{su}(N), \quad \text{so}(N) \]

A semi-simple algebra is the direct sum of simple algebras.

\[ \text{e.g.} \quad \text{su}(N) = \text{su}(3) \oplus \text{su}(2) \oplus \text{u}(1) \]

**Theorem:** all finite-dimensional reps. of semi-simple algebras are Hermitian. You can construct theories where the symmetry is a unitary transformation on fields

\[ \text{[unitary theories concern probabilities]} \]

We are also interested in the case where the number of generators is finite.

\[ \rightarrow \text{compact algebras (because the corresponding Lie group is a finite-dim compact manifold)} \]

The requirement of being simple and compact is very stringent: classification.

- **Unitary groups:** \( U(N) \)

  defining reps.
  acts on space of \( N \)-dim complex vectors

  then
  \[ U^\dagger U = (e^{\hat{n}aT})^\dagger (e^{\hat{n}aT}) = e^{-\hat{n}aT + \hat{n}aT} = I \]

  then a complex inner product is preserved: true \( \psi, \chi \) states
  \[ (\psi^\dagger \chi)^\dagger = \psi^\dagger \chi = \psi^\dagger \chi \]

  with \( T \) Hermitian
  \[ T^\dagger = T \]

  \[ \hat{n}aT \quad \text{with} \quad T \quad \text{Hermitian} \]

  \[ e^{-\hat{n}aT + \hat{n}aT} = I \]

  \[ (\psi^\dagger \chi)^\dagger = \psi^\dagger \chi = \psi^\dagger \chi \]
now $U(N)$ contains the pure phase transformations

$$U = \exp i \theta,$$

which form a $U(1)$ subgroup

there are removed by requiring $\det U = 1$ (and not a complex phase
with $|1| = 1$)

which gives the simple group $SU(N)$

where dimension is $d(SU(N)) = N^2 - 1$

because $N \times N$ complex matrix

\[ UTU^+ = 1 \]

\[ \det U = 1 \]

\[ \Rightarrow 2N^2 - N^2 - 1 = N^2 - 1. \]

- **ORTHOGONAL GROUPS** preserve a real inner product

  $\psi, \chi$ vectors

  $O \psi \cdot O \chi = \psi^T O^T O \chi$

  $O^T O = I$

  how many generators?

  real $N \times N$ matrix satisfying

  $O^T O = 1$

  so $N + \frac{N^2 - N}{2}$ independent conditions

  \[ d(O(N)) = N^2 - \left[ N + \frac{N^2 - N}{2} \right] \]

  \[ = \frac{N(N-1)}{2} \]

  now $\det O = \pm 1$ $\Rightarrow$ take $+1$ to get $SO(N)$

- **SYMPLECTIC GROUPS** $Sp(N) \ N$ even

  preserve an antisymmetric inner product

  \[ \psi^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \chi \]

  \[ \text{in} \ \frac{N}{2} \times \frac{N}{2} \text{ block form} \]

  $S^T J S = J$

  \[ d(Sp(N)) = N^2 - \left[ \frac{N^2 - N}{2} \right] \]

  \[ = \frac{N(N+1)}{2} \]

  note this group in $N$ dimensions

  then clear that

  $d(SO(N)) = \frac{N(N-1)}{2}$:

  number of planes in $N$ dim

  number of conditions

  from $S^T J S = J$

  antisymmetric since

  \[ (S^T J S)^T = S^T J^T S = -S^T J S \]
\[ \text{Exceptional groups (5 of them): } G_2, F_4, E_6, E_7, E_8 \]

\[ [SU(N) \text{ has } n = N-1] \]

\[ \text{Rank of the group} \]

\[ \text{Dimension of Cartan subgroup, which is the maximal commuting subgroup; } \]

\[ \text{Equivalently, \# of diag generators} \]

This completes clarification of compact simple Lie algebras.

Note also that if there is any Hermitian repr. (as for nonsimple algebras) then the structure constants are REAL:

\[ [T^a, T^b] = \hat{n} \delta^{abc} T^c \implies [T^a, T^b]^T = -\hat{n} \delta^{abc} T_c \]

\[ (T^a, T^b - T^b, T^c) \]

\[ [T^b, T^c] = \hat{n} \delta^{abc} T^c \]

\[ \delta_{abc} \text{ is real and} \]

\[ \text{so} \]

\[ \text{the 2 indices} \]

\[ \delta_{abc} = \delta_{abc}^* \]

---

**General Properties of Repr**

Reprs. of algebra are constructed by embedding generators into matrices. One can show that for compact semi-simple Lie algebras, for any repr. \( R \) we can choose a basis for the generators such that

\[ T_R (T^a_R T^b_R) = T(R) \delta^{ab} \]

now combined with \([T^a_R, T^b_R] = \hat{n} \delta^{abc} T^c_R \]

this implies

\[ T_n [T^a_R, T^b_R] T^n_c = \hat{n} \delta^{abd} T_n T^n_d T^n_c \]

\[ = \hat{n} \delta^{abc} T(R) \delta^{dc} = \hat{n} T(R) \delta^{abc} \]

\[ \implies \delta_{abc} = -\frac{\hat{n}}{T(R)} T_n [T^a_R, T^b_R] T^n_c \]
Now this implies that \( \delta^{a}{}_{b}{}^{c} = \delta^{b}{}_{c}{}^{a} \) because

\[
T_{n} \left[ T_{n}^{a} T_{n}^{b} T_{n}^{c} \right] = T_{n} \left[ T_{n}^{a} T_{n}^{b} T_{n}^{c} - T_{n}^{b} T_{n}^{a} T_{n}^{c} \right]
\]

by cyclicity of trace.

Now recall that \( \delta^{a}{}_{b}{}^{c} = -\delta^{b}{}_{c}{}^{a} \) combined, then tell us that \( \delta^{a}{}_{b}{}^{c} \) are completely antisymmetric.

Now for a given \( R \), infinitesimal transform under the group is

\[
\phi \rightarrow (1 + \eta a \cdot T_{n}^{a}) \phi \quad \text{hence complex conj}
\]

\[
\phi^{*} \rightarrow (1 - \eta a \cdot T_{n}^{a}^{*} \phi^{*} \quad \text{but also we can define the conjugate rep.}
\]

\[
\phi^{*} \rightarrow (1 + \eta a \cdot T_{n}^{a}) \phi
\]

hence

\[
T_{n}^{a} = - (T_{n}^{a})^{*} = - (T_{n}^{a})^{T}.
\]

Now \( R \) is equivalent to \( R \), if there exists a unitary transform \( U \) such that

\[
T_{n}^{a} = U T_{n}^{a} U^{T}.
\]

In this case, \( R \) is a \textbf{real} rep.

Then we can always find a matrix \( E_{a}{}^{b} \) such that if \( \psi, \chi \in R \) then \( \psi_{a} E_{a}{}^{b} \chi_{b} \) is \textbf{invariant}

If \( E_{a}{}^{b} \) \textbf{SYM} \textbf{strictly real}

\textbf{ANTISYM} \textbf{PSEUDOREAL}
Reprs. of SU(N) Groups

Free theory of N complex fields is automatically invariant under \( U(N) \cong SU(N) \times U(1) \) \( \rightarrow \) hence the importance of \( SU(N) \) groups.

The two most important representations are: FUNDAMENTAL and ADJOINT.

- FUNDAMENTAL: acts on space of \( N \)-dim complex vectors. Smallest non-trivial repr. of the algebra. Dimension is \( N \).
  - For \( SU(N) \), set of \( N \times N \) Hermitian, traceless matrices.

\[
U(\mathbb{C}) = e^{-i \mathbb{H} \cdot \frac{\pi}{2}} \quad \text{and diagonalize} \quad V \mathbb{H} \cdot \frac{\pi}{2} U^{-1} = D
\]

(\( \text{Hermitian matrices are diagonalizable} \))

\[
\det U = \det e^{i \mathbb{D}} = \prod_i e^{i \mathbb{D}_{ii}} = e^{i \sum_i \mathbb{D}_{ii}} = e^{i \text{Tr} D}
\]

\( \det \) is basis-indep

\( \text{Tr} \) is basis-indep

\[
\det U = 1. \text{ Note that traceless traceless elements argument applies for any semisimple algebra.}
\]

- For \( SU(2) \), \( T^a = \frac{\sigma^a}{2} \) \( \sigma^a \) are Pauli matrices

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

\( \sigma^a \) satisfy

\[
[T^a, T^b] = \frac{1}{4} [\sigma^a, \sigma^b]
\]

\[
= \frac{1}{4} 2\mathbb{\hat{\sigma}} \epsilon^{abc} \sigma^c = \mathbb{\hat{\sigma}} \epsilon^{abc} T^c
\]

(structure constants)

- For \( SU(3) \), \( T^a = \frac{\lambda^a}{2} \) \( \lambda^a \) are Gell-Mann matrices
Now for the 2 of \( SU(2) \) we have

\[
T^a_R = \frac{\sigma^a}{2} \quad T^a_i = -\frac{\sigma^a}{2}
\]

but we know that

\[
\sigma^a \sigma^b \sigma^c = -\sigma^a
\]

hence

\[
T^a_i = -\frac{\sigma^a}{2} = \sigma^i \left( \frac{\sigma^a}{2} \right) \sigma^a = \sigma^i T^a_{\sigma^a} \sigma^a
\]

i.e., REAL repr. with \( U = \sigma^2 \),

\[
U^\dagger = U
\]

also, \( E_{ab} = (\hbar \sigma^2)_{ab} = E_{ab} \) [Exercise]

so it is pseudo-real

For \( N > 2 \), the \( N \) of \( SU(N) \) is instead complex

(so in part. \( 3 \neq \overline{3} \) for \( SU(3) \))

- **Adjoint**: acts on the space spanned by the generators themselves

\[
(T_{adj}^a)_{bc} = q \varepsilon^{abc}
\]

(\( q \) purely Im)

\[
\text{dimension is } N^2 - 1
\]

(number of generators)

E.g. for \( SU(2) \):

\[
(T_{adj}^a)_{bc} = q \varepsilon^{abc} \quad \rightarrow T_{adj}^a = q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \text{ etc.}
\]

**Exercise**: find \( q \) such that \( T_{adj}^a \) satisfy comm. relations of algebra.

Adjoint repr. is important because it is the repr. where non-Abelian gauge fields transform.

Note that the conjugate repr. of the adjoint is

\[
T_{adj}^a = -(T_{adj}^a)^* = - (q \varepsilon^{abc})^* = +q \varepsilon^{abc} = T_{adj}^a
\]

hence the adjoint is always real \( q \in \text{Im} \)

\[
\varepsilon^{abc} \text{ real}
\]

\( (U = 1) \).
CASIMIR OPERATOR

For any repr. $T^a$, we know that $T^2 \equiv T^a T^a$ (sum over $a$) commutes with all the generators:

$[T^2, T^b] = [T^a T^a, T^b]$

$= T^a [T^a, T^b] + [T^a, T^b] T^a$

$= T^a \delta^{abc} T^c + \delta^{abc} T^a T^c$

$= \delta^{abc} \{ T^a, T^c \}$

Hence $T^2$ must be proportional to the identity, by Schur's lemma. So

$T^a T^a = C^2(R) I$

$\uparrow$ quadratic casimir of $R$

[Example: $SU(2)$, $J^2 = T^a T^a$ is commutative with eigenvectors $\frac{j(j+1)}{2}$ (total spin)]

As we discussed, we also have for an appropriate choice of basis

$T^a (T^a T^b) = T(R) \delta^{ab}$

Now contract this with $\delta^{ab}$

$\uparrow$ index of $R$

(sometimes called $c(R)$)

$T^a (T^a T^a) = T(R) d(G)$

From $\uparrow$

$C^2(R) d(R) \quad \Rightarrow \quad d(R) C^2(R) = T(R) d(G)$

which is useful to compute $C^2(R)$

- **Fundamental of $SU(N)$**, typical to choose $T(\text{fund}) = \frac{1}{2}$, or in the case of $SU(2)$

Then

$C^2(\text{fund}) = \frac{T(\text{fund}) d(G)}{d(\text{fund})} = \frac{1}{2} \frac{N^2-1}{N} = C_F$

$\uparrow$ often used
ADJOINT of SU(N)

First note that $d(G) = d(G)$, no $C_2(adj) = T(adj)$.

How do we compute $C_2(adj)$? Instructive to do it by building adjoint as product of fund and antiqui: given 2 reprs, $n_1$ and $n_2$, their direct product is a repr of dim. $d(n_1) \cdot d(n_2)$. Objects transforming under it are tensors $\sum_{pq} p \in n_1, q \in n_2$. The product repr. can in general be decomposed on direct sum of irreps: $n_1 \times n_2 = \sum \ n_i$.

The matrices are $t_{n_1,n_2}^a = t_{n_1}^a \otimes 1 + 1 \otimes t_{n_2}^a$.

Now the quad. form of the product is

$$t_{n_1,n_2}^a t_{n_1,n_2}^b = (t_{n_1}^a)^2 \otimes 1 + 1 \otimes (t_{n_2}^a)^2 + 2 t_{n_1}^a \otimes t_{n_2}^a$$

take the trace

$$T_n (t_{n_1,n_2}^a) = T_n (t_{n_1}^a) \cdot d(n_2) + d(n_1) \cdot T_n (t_{n_2}^a) + 0$$

$$= \sum_{n_1} C_2(n_1) \cdot d(n_1) \cdot d(n_2) + d(n_1) \cdot C_2(n_2) \cdot d(n_1)$$

$$= [C_2(n_1) + C_2(n_2)] \cdot d(n_1) \cdot d(n_2)$$

But also, from $\nabla$ we get

$$T_n (t_{n_1,n_2}^a) = \sum_{n_1} T_n (t_{n_1}^a) = \sum_{n_1} C_2(n_1) \cdot d(n_1)$$

Now apply to case of $n_1 = N$, $n_2 = \overline{N}$ in SU(N):

$$N \times \overline{N} = 1 + (N^2 - 1)$$

Then from $\nabla = \nabla$ get

$$[C_2(N) + C_2(\overline{N})] \cdot N^2 = C_2(adj) \cdot d(adj)$$

$$= 2 C_2(N) N^2 = C_2(adj) (N^2 - 1)$$

$$\Rightarrow C_2(adj) = 2N^2 \cdot \frac{N^2 - 1}{2N} \cdot \frac{1}{N^2 - 1} = N = C_A$$

Hence $C_2(adj) = C_A = N$. Therefore $T(adj) = T_A = N$. 